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Propagation of TE_{01} Waves in Curved Wave Guides

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TE_{01} waves transmitted through curve wave guides lose power by conversion to other modes, especially to TM_{11}.

This power transfer to coupled modes is explained by the theory of coupled transmission lines. It is shown that the power interchange between coupled lines and their propagation constants can be derived from a single coupling discriminant.

Earlier calculations of TE_{01} conversion loss in circular wave guide bends are confirmed and extended to S-shaped bends.

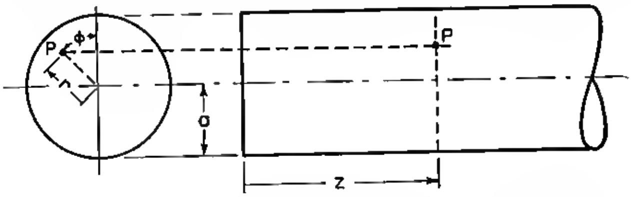
Tolerance limits for random deflections from an average straight course are given.

THE TE_{01} mode of propagation in circular wave guides has great potential value for the transmission of wide-band signals because its attenuation decreases with frequency. In order to take full advantage of this property one must use sufficiently large wave guides to operate well above the cutoff of the lowest transmitted frequency. The difficulty of this transmission method lies in the fact that TE_{01} is not the dominant mode and that energy may be lost by transfer to the many other modes capable of transmission in the wave guide. In an ideal wave guide, which is perfectly straight, perfectly circular and perfectly conducting, the propagation is undisturbed; but slight imperfections and especially a slight curvature of the wave guide axis may produce serious disturbances.

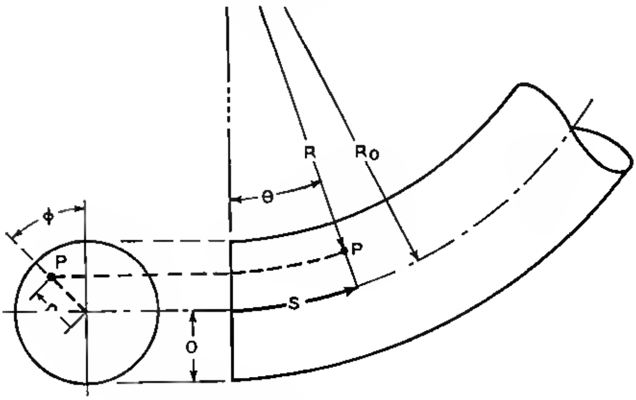
The character of these disturbances has been investigated in several publications by Prof. M. Jouguet[1] and in unpublished work by Mr. S. O. Rice of the Bell Telephone Laboratories. Both Jouguet and Rice use the method of perturbations, which is a form of calculus invented by astronomers to compute the deviations from the exact elliptical orbits of the planets which are caused by the disturbing influences of their fellow planets. Although the above-mentioned authors obtained valuable results, the interpretation of their solutions is difficult due to this rather abstract mathematical formulation. To most engineers the understanding of a physical problem is greatly helped if it is possible to use a method of analysis which is elementary in character and easily interpreted in familiar physical terms. The familiar concept on which the present treatment will be based is that of coupled circuits.

[1] See References 2 and 3, listed on page 7.

It has been stated by the earlier authors that the curvature of the wave guide produces a coupling between modes. Before going into a detailed analysis one may estimate by inspection the nature of this coupling and the kind of modes that are most strongly coupled to each other. Figure 1a shows the cross section and the longitudinal section of a straight cylindrical wave guide. The location of every point inside the wave guide is determined by three coordinates: the radial distance r from the cylinder axis; the azimuth angle φ from an arbitrary 0 line and the axial distance z from the



a - CYLINDRICAL CO-ORDINATES IN STRAIGHT WAVEGUIDE



b - TOROIDAL CO-ORDINATES IN CURVED WAVEGUIDE

Fig. 1

origin. If the wave guide is bent as shown on Fig. 1b, but a wave front at right angles to the cylinder axis is to be maintained, the waves must be shortened at the inside of the bend and lengthened at the outside of the bend. Regarding compression as a positive and expansion as a negative deformation, one sees that the distortion of the wave shape is proportional to the curvature of the wave guide multiplied by the cosine of the azimuth angle. It is natural to assume that the coupling between modes is proportional to this distortion.

Now it is known that all modes of propagation in a circular wave guide

can be derived from functions $J_n(\chi r)$ cos $n\varphi$. In these functions, n is called the aximuthal index because it indicates the type of symmetry around the circumference of the wave guide. When these characteristic functions are multiplied by the distortion factor cosine φ, the resulting expressions are proportional to the sum of cosine $(n + 1)\,\varphi$ and cosine $(n - 1)\,\varphi$. This means that the bending of the wave guide couples mainly those modes which differ by ± 1 in azimuth index. Since the TE_{01} mode has the azimuthal index 0, it is coupled to all modes of the type TE_{1m} and TM_{1m}.

In the above qualitative discussion we have claimed that coupling exists without defining the physical coupling parameters and their effects. We must now supply this definition and show that the TE_{01} mode is particularly susceptible to coupling losses.

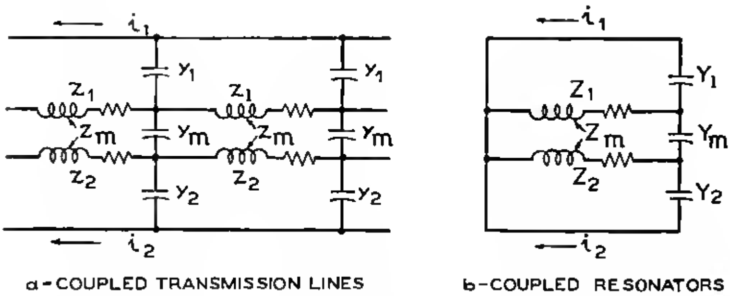


Fig. 2

Our investigation is guided by S. A. Schelkunoff's statement[2] that a wave guide mode has the same equation of propagation as a high-pass transmission line. Schelkunoff further points out[3] that the high-pass character of circular wave guide modes can be interpreted as the effect of interfering plane waves whose directions of propagation deviate from the wave guide axis by a constant slanting angle.

We therefore approach the problem of coupled wave guide modes by studying the hehavior of two coupled transmission lines such as shown on Fig. 2a. Each transmission line is schematically shown as an array of small ladder sections. The series impedances per unit length of the lines are z_1 and z_2; their shunt admittances per unit length, y_1 and y_2. The two lines are loosely coupled by small mutual series impedances per unit length (z_m) and by small mutual shunt admittances per unit length (y_m).

A network of coupled ladder sections is more tractable than a wave guide structure, but still somewhat complicated. Let us therefore carry the analogy one step further. Figure 2b shows two resonant circuits, each

[2] Ref. 4, pp. 378 and 381 of the book.
[3] Ref. 4, p. 410 of the book.

consisting of single capacity C, and an impedance Z which includes an inductance L and a damping resistance R. The resonators are coupled by a small mutual inductance Z_m and by a small mutual capacity Y_m.

The behavior of coupled resonators is very well known to radio engineers. They occur as tuned transformers in amplifier circuits, as band-pass filters and as "tank circuits" in radio transmitters. Even before the advent of radio, their acoustical equivalents were studied in the form of resonant tuning forks. The mathematical aspects of this problem were already clearly set forth in a paper by Wien written in 1897[4]. He showed that the interaction between the free vibrations of two tuned circuits depends on the coupling coefficient and on the ratio of their complex resonance frequencies. The closer the two frequencies are to each other, the less coupling is needed to transfer energy between the two circuits. The reason is that the individual free vibrations of two nearly synchronous circuits remain in step long enough to accumulate the small energy transfer impulses of many vibrations.

Now consider the two transmission lines of Fig. 2a and assume that a constant frequency signal is impressed upon the input of one or both of them. The signals are carried along the two lines as traveling waves. Again it is true that loosely coupled signals affect each other strongly if they remain in step. With traveling waves "remaining in step" means that they must travel with approximately equal phase velocities. We conclude that the phase velocities or phase constants of coupled transmission lines play a similar role as the resonant frequencies of coupled tuned circuits. This intuitive reasoning is confirmed by analysis (see Section 1 of the analytical part of this paper).

We thus find that we must expect trouble for TE_{01} wave guide transmission if a mode with an azimuth index 1 has a propagation constant close to that of the TE_{01}. It so happens that there exists one mode, the TM_{11}, which in an ideal wave guide has exactly the same propagation constant as the TE_{01}. This then should be the principal source of trouble—and from previous work it is known that such is the case.

Our discussion of coupled transmission lines has shown that the interaction effects are functions of their relative uncoupled propagation constants and of the coupling coefficient. The propagation constants of the TE_{01} and TM_{11} wave guide modes are known but their coupling coefficient remains to be found.

Since the energy of the transmission modes is located in the dielectric inside the wave guide, we consider first the coupling between the plane "slant wave" groups from which the modes are built up.

[4] Reference 5.

As shown in the analytical part, the coupling coefficient of these slant waves may be defined as the energy interchanged between the modes per unit length of line divided by the geometric mean of the energies per unit length stored up in each of the modes.

From the coupling coefficient of the slant waves the coupling coefficient of the wave guide modes is derived.

On the basis of the above physical interpretation the analysis is carried out and the properties of TE_{01} propagation through curved wave guides of various shapes are derived in the analytical part of this paper which is subdivided into the following nine sections:

Section 1 develops an approximate theory of loosely coupled, weakly damped circuits. The theory is first derived for coupled resonators which are familiar to communication engineers, and then applied in similar form to coupled transmission lines. It is shown that the important interaction properties of coupled lines are functions of a single coupling discriminant. The relative energy content of the two lines in each of the two possible coupled modes is plotted as a function of the coupling discriminant.

Section 2 contains the field equations of a straight circular wave guide and their modification by a toroidal bend.

Section 3 gives the solutions of the field equations for the uncoupled TE_{01} and TM_{11} modes in wave guides with infinite, and with small but finite conductivity.

Section 4 applies the coupling theory to the TE_{01} and TM_{11} modes in circular wave guide bends. The coupling coefficient, coupling discriminant and energy division between the two modes are derived as functions of the wave guide diameter bending radius and conductivity and of the signal frequency.

Section 5 derives the critical bending radius and the attenuation of TE_{01} waves in long wave guides of constant curvature. Two numerical examples are given.

Section 6 shows that in a curved section of wave guide which follows a long straight section or other source of pure TE_{01} the energy fluctuates back and forth between a condition of pure TE_{01} and of predominant TM_{11}. The length and magnitude of the fluctuations are derived.

Section 7 computes the increase in average attenuation caused by serpentine bends of regular shapes. Numerical examples are tabulated.

Section 8 shows that the results of Section 7 can be applied to helical bends and to small two-dimensional random deviations from a straight course.

Section 9 shows that for any given statistical distribution of random angular deviations the average attenuation is minimized by an optimum

wave guide radius for each signal wave length and by an optimum signal wave length for each wave guide radius.

Numerical examples are given for sinusoidal bends.

Summary of Results

1. The energy loss of TE_{01} waves in curved wave guides by conversion into the TM_{11} mode is interpreted as a case of coupling between resonant transmission lines.
2. In a pair of coupled lines the energy cannot be confined entirely to a single line but travels through both in one or both of two possible combination modes.
3. All important properties of coupled circuits, including wave guide modes, are functions of a single discriminant.
4. When the discriminant is much smaller than one, most of the energy can be carried in one line or component mode.
5. When the discriminant is much larger than one, the energy flow is nearly equally divided between the two lines or component modes.
6. In wave guides of typical dimensions the coupling discriminant becomes one for a "critical" bending radius greater than a mile. For all sharper bends, that is for most practical installations, the discriminant is greater than one.
7. In a long wave guide section with more than critical curvature the average attenuation constant is the arithmetic mean between those of the TE_{01} and the TM_{11} modes.
8. If a wave guide region carrying pure TE_{01} is followed by a curved region, the energy in the curved region fluctuates back and forth between pure TE_{01} and predominant TM_{11}. The location of TE_{01} minima and maxima is a function of the signal frequency, the wave guide diameter and the total bending angle.
9. For highly supercritical curvatures the bending angles at which minima and maxima occur are nearly independent of the curvature and approach the limiting values previously computed by Jouguet and Rice. The minima approach zero. When the bending radius approaches or exceeds the critical value, the maxima and minima become shallower and their spacing is increased by a function of the coupling discriminant.
10. For regular serpentine bends or random angular deviations from an average straight course which are much smaller than the first extinction angle, the percentage increase in average attenuation is proportional to the square of the maximum deviation and to the fourth power of wave guide diameter and signal frequency.
11. Wave guide installations of practical dimensions for frequencies now

attainable are tolerant to random angular deviations of the order of 1 degree.

12. For any expected distribution of random angular deviations there exists an optimum wave guide radius for each signal wave length and an optimum signal wave length for each wave guide radius, which minimize the average attenuation.

References

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4. S. A. Schelkunoff, Electromagnetic Waves, D. Van Nostrand Company, Inc., New York, 1943.
5. M. Wien, Ueber die Rueckwirkung eines resonierenden Systems, *Ann. d. Physik*, 1897, Vol. 61, pp. 151–189.

ANALYSIS

1. Interaction of Coupled Circuits

1.1 *Free Oscillations of Coupled Resonators (Fig. 1B)*

The circuits are coupled according to the following four equations:

$$e_1 = -Z_1 i_1 + Z_m i_2 \quad 1.1\text{-}1$$

$$i_1 = Y_1 e_1 + Y_m e_2 \quad 1.1\text{-}2$$

$$e_2 = -Z_2 i_2 + Z_m i_1 \quad 1.1\text{-}3$$

$$i_2 = Y_2 e_2 + Y_m e_1 \quad 1.1\text{-}4$$

where index $_1$ refers to circuit 1, index $_2$ to circuit 2 and index $_m$ to the mutual coupling impedance and admittance. The coupled oscillations have the solution:

$$e_1 = E_{1a}\epsilon^{p_a t} + E_{1b}\epsilon^{p_b t} \quad 1.1\text{-}5$$

$$e_2 = E_{2a}\epsilon^{p_a t} + E_{2b}\epsilon^{p_b t} \quad 1.1\text{-}6$$

In the limiting case of zero coupling ($Y_m = 0, Z_m = 0$) the obvious solution shows independent oscillations in the two separate circuits:

$$e_{10} = K_1 i_{10} = E_{10}\epsilon^{p_1 t} \quad 1.1\text{-}7$$

$$e_{20} = K_2 i_{20} = E_{20}\epsilon^{p_2 t} \quad 1.1\text{-}8$$

The wave impedance K_1 of the primary circuit is found by dividing equation 1.1-1 by 1.1-2

$$K_1 = \frac{e_1}{i_1} = \sqrt{\frac{Z_1}{Y_1}}$$

Similarly,

$$K_2 = \frac{e_2}{i_2} = \sqrt{\frac{-Z_2}{Y_2}}$$

By multiplying equation 1.1-1 by 1.1-2 one finds

$$-Z_1Y_1 = 1 \qquad 1.1\text{-}9$$

from which one can compute the exponent p_1. In the specific circuits shown in Fig. 1b

$$Z_1 = L_1p_1 + R_1 \quad \text{and} \qquad 1.1\text{-}10$$

$$Y_1 = C_1p_1 \qquad 1.1\text{-}11$$

From 1.1-9, 10 and 11

$$p_1 = -\delta_1 + j\omega_1 = -\frac{R_1}{2L_1} + j\sqrt{\frac{1}{L_1C_1} - \frac{R_1^2}{4L_1^2}} \qquad 1.1\text{-}12$$

and by analogy

$$p_2 = -\delta_2 + j\omega_2 = -\frac{R_2}{2L_2} + j\sqrt{\frac{1}{L_2C_2} - \frac{R_2^2}{4L_2^2}} \qquad 1.1\text{-}13$$

In equations 1.1-7 and 1.1-8, E_{10} and E_{20} are amplitude constants determined by boundary conditions. In equations 1.1-12 and 1.1-13, δ_1 and δ_2 are the decay or damping constants, ω_1 and ω_2 the radian frequencies.

With finite but loose coupling and small damping the circuits can oscillate with either or both of the two frequencies.

$$p_a = \frac{p_1 + p_2}{2} + \frac{p_1 - p_2}{2}\sqrt{1 + \kappa^2} = p_1 + 0.5\, p_2(1 - \sqrt{1 + \kappa^2}) \quad 1.1\text{-}14$$

$$p_b = \frac{p_1 - p_2}{2} - \frac{p_1 - p_2}{2}\sqrt{1 + \kappa^2} = p_2 + 0.5\, p_1(1 - \sqrt{1 + \kappa^2}) \quad 1.1\text{-}15$$

In the last two equations, the symbol κ, defined by $\kappa = \sqrt{\dfrac{p_1p_2}{p_1 - p_2}} \cdot k$, may be called the coupling discriminant. The first term of the product on the right side of this expression is the reciprocal of the fractional difference between the uncoupled frequencies; the second term k is the "coupling coefficient."

When there is only one coupling impedance, the coupling coefficient is usually defined as the mutual circuit impedance divided by the geometric mean of the separate circuit impedances. A broader definition which applies to all combinations of mutual impedances and admittances is

$$k = \frac{P_{12}}{\sqrt{P_1P_2}} = \frac{P_{21}}{\sqrt{P_1P_2}} \qquad 1.1\text{-}16$$

In this equation P_1 is the energy stored in circuit 1, P_2 the energy stored in circuit 2 and P_{12} is the energy transferred from one circuit to the other. One finds

$$P_1 = \frac{e_1^2}{2K} + \frac{i_1^2 K}{2} = \frac{e_1^2}{K_1} = i_1^2 K_1 \tag{1.1-17}$$

$$P_2 = \frac{e_2^2}{K_2} = i_2^2 K_2 \tag{1.1-18}$$

$$P_{12} = \frac{e_{12} e_2}{K_2} = i_{12} i_2 K_2 = \frac{e_{21} e_1}{K_1} + i_{21} i_1 K_1 \tag{1.1-19}$$

Equations 1.1-5 and 1.1-6 contain four amplitude constants. Two of these, for instance E_{1a} and E_{2b}, can be adjusted to satisfy boundary conditions. The other two are fixed by the equation

$$\frac{E_{2a}^2 K_1}{E_{1a}^2 K_2} = \frac{p_a - p_1}{p_a - p_2} = \frac{p_b - p_2}{p_b - p_1} = \frac{E_{1b}^2 K_2}{E_{2b}^2 K_1}$$

The square root of this expression,

$$\frac{E_{2a}}{E_{1a}} \sqrt{\frac{K_1}{K_2}} = \frac{E_{1b}}{E_{2b}} \sqrt{\frac{K_2}{K_1}} = A$$

may be called the *normalized amplitude ratio.* It is a vector quantity denoting the amplitude ratio and phase relation of each oscillation frequency in the two circuits, assuming that they have been normalized to equal resistances by an ideal transformer. The absolute value

$$\left| \frac{E_{2a}^2 K_1}{E_{1a}^2 K_2} \right| = W_a = \frac{1}{W_b}$$

is the ratio of the energies stored in the two circuits oscillating at frequencies p_a and p_b respectively.

From 1.1-14, 15, 16 and 18

$$W_a = \left| \frac{\sqrt{1 + \kappa^2} - 1}{\sqrt{1 + \kappa^2} + 1} \right| = W$$

$$A_a = \sqrt{1 + \kappa^{-2}} - \kappa^{-1} = A$$

When the indexes are left off, $W < 1$ and $|A| < 1$ by definition. One sees that energy, amplitude and phase relations between the coupled circuits at each oscillating frequency are governed by the coupling discriminant. This also applies to the damping coefficients and frequencies

of the coupled oscillations. It can be shown by combining and transforming equations 1.1-14, 15, 19, that the coupled damping coefficients are

$$\delta_a = \frac{\delta_1 + \delta_2 W}{1 + W} = \delta_1 \frac{W_{1a}}{W_{\text{total}}} + \delta_2 \frac{W_{2a}}{W_{\text{total}}}$$

$$\delta_b = \frac{\delta_1 W + \delta_2}{1 + W} = \frac{\delta_1 W_{1b}}{W_{\text{total}}} + \frac{\delta_2 W_{2b}}{W_{\text{total}}}$$

The damping constants of two coupled resonances are found by combining the uncoupled damping constants in the same proportion as the energies oscillating in the two resonators.

The coupled frequencies are

$$\omega_a = \frac{\omega_1 - W_{\omega 2}}{1 - W} \text{ and}$$

$$\omega_b = \frac{\omega_2 - W_{\omega 1}}{1 - W}$$

1.2 *Forced traveling waves in coupled transmission lines (Fig. 1A).*

The two lines are coupled according to the four equations

$$\Gamma e_1 = z_1 i_1 + z_m i_2$$

$$\Gamma i_1 = y_1 e_1 + y_m e_2$$

$$\Gamma e_2 = z_2 i_2 + z_m i_1$$

$$\Gamma i_2 = y_2 e_2 + y_m e_1$$

which may be compared to the corresponding equations of section 1.1. There is a dimensional difference because in transmission lines the series impedances z are measured in ohm/meter and the shunt reactances y in mho/meter. Γ is the propagation constant of the wave traveling in the $+s$ direction. If a sinusoidal signal with the radian frequency ω is impressed upon the input of the lines, the coupled waves have the solution

$$e_1 = E_{1a} \epsilon^{j\omega t - \Gamma_a s} + E_{1b} \epsilon^{j\omega t - \Gamma_b s} \quad 1.2\text{-}1$$

$$e_2 = E_{2a} \epsilon^{j\omega t - \Gamma_a s} + E_{2b} \epsilon^{j\omega t - \Gamma_b s} \quad 1.2\text{-}2$$

For zero coupling one finds, in analogy to Section 1.1

$$e_{10} = K_1 i_1 = E_{10} \epsilon^{j\omega t - \Gamma_1 s}$$

$$e_{20} = K_2 i_2 = E_{20} \epsilon^{j\omega t - \Gamma_2 s}$$

it

$$K_1 = \sqrt{\frac{z_1}{y_1}}$$

$$K_2 = \sqrt{\frac{z_2}{y_2}}$$

and

$$\Gamma_1 = \sqrt{y_1 z_1}$$
$$\Gamma_2 = \sqrt{y_2 z_2}$$

E_{10} and E_{20} are independent integration constants. For finite but loose coupling and small attenuation constants one finds in analogy to 1.1-14 and 1.1-15

$$\Gamma_a = \frac{\Gamma_1 + \Gamma_2}{2} + \frac{\Gamma_1 - \Gamma_2}{2}\sqrt{1 + \kappa^2} = \Gamma_1 + 0.5\,\Gamma_2(1 - \sqrt{1 + \kappa^2})$$

$$\Gamma_b = \frac{\Gamma_1 + \Gamma_2}{2} - \frac{\Gamma_1 - \Gamma_2}{2}\sqrt{1 + \kappa^2} = \Gamma_2 + 0.5\,\Gamma_1(1 - \sqrt{1 + \kappa^2})$$

where

$$\kappa = \frac{k}{\Gamma_1 - \Gamma_2}\sqrt{\Gamma_1 \Gamma_2} \qquad 1.2\text{-}3$$

is the coupling discriminant. Just as in Section 1.1 the coupling coefficient k is defined by the equation

$$k = \frac{P_{12}}{\sqrt{P_1 P_2}} = \frac{P_{21}}{\sqrt{P_1 P_2}} \qquad 1.2\text{-}4$$

P_1 is the energy per unit length stored in line 1; P_2, the energy per unit length stored in line 2, and $P_{12} = P_{21}$, the energy per unit length interchanged between the lines. The waves can travel in the coupled lines with either or both of two transmission constants. Two of the amplitude vectors in equations 1.2-1 and 1.2-2, for instance E_{1a} and E_{2b}, are free to satisfy boundary conditions; the other two are determined by the equation

$$\frac{E_{2a}^2 K_1}{E_{1a}^2 K_2} = \frac{\Gamma_a - \Gamma_1}{\Gamma_a - \Gamma_2} = \frac{\Gamma_b - \Gamma_2}{\Gamma_b - \Gamma_1} = \frac{E_{1b}^2 K_2}{E_{2b}^2 K_1}$$

$$\frac{E_{2a}}{E_{1a}}\sqrt{\frac{K_1}{K_2}} = \frac{E_{1b}}{E_{2b}}\sqrt{\frac{K_2}{K_1}} = A_a = -\frac{1}{A_b} \qquad 1.2\text{-}5$$

A is the *normalized amplitude and phase ratio* for two lines transformed to equal wave impedances.

$$\left|\frac{E_{2a}^2 K_1}{E_{1a}^2 K_2}\right| = W_a = \frac{1}{W_b}$$

$$W_a = \left|\frac{\sqrt{1 + \kappa^2} - 1}{\sqrt{1 + \kappa^2} + 1}\right| = W$$

is the ratio of energy flow in the two lines. At the propagation constant Γ_a,

$$A_a = \sqrt{1 + \kappa^{-2}} - \kappa^{-1} = A \qquad 1.2\text{-}6$$

When the indexes are left off, $W < 1$ and $|A| < 1$ by definition.
In a manner analogous to that of Section 1.1 it can be shown that the coupled attenuation constants are

$$\alpha_a = \frac{\alpha_1 + \alpha_2 W}{1 + W} = \alpha_1 \frac{W_{1a}}{W_{\text{total}}} + \alpha_2 \frac{W_{2a}}{W_{\text{total}}} \qquad 1.2\text{-}7$$

$$\alpha_b = \frac{\alpha_2 + \alpha_1 W}{1 + W} = \alpha_1 \frac{W_{1b}}{W_{\text{total}}} + \alpha_2 \frac{W_{2b}}{W_{\text{total}}} \qquad 1.2\text{-}8$$

The attenuation constants of the coupled waves are found by combining the uncoupled attenuation constants in the same proportion as the energies traveling in the two lines.
The coupled phase constants are

$$\beta_a = \frac{\beta_1 - W\beta_2}{1 - W} \text{ and} \qquad 1.2\text{-}9a$$

$$\beta_b = \frac{\beta_2 - W\beta_1}{1 - W} \qquad 1.2\text{-}9b$$

From equations 1.2-5 to 1.2-8 one sees that the coupled propagation constants are conveniently described in terms of the power ratio W. W itself is a known function of the complex coupling determinant κ which is shown on the attached Fig. 4 for the following three special cases:

Case 1.

The two lines have *equal phase* constants and *different attenuation* constants: $\beta_2 = \beta_1 \qquad \alpha_2 \gtrless \alpha_1$

κ is an imaginary number.

W changes its character abruptly at the critical coupling.

$|\kappa \text{ critical}| = 1$

For $|\kappa| < 1$

$$W < 1; \qquad \alpha_b \gtrless \alpha_a; \qquad \beta_b = \beta_a$$

For $|\kappa| \geqq 1$

$$W = 1; \qquad \alpha_b = \alpha_a; \qquad \beta_b \gtrless \beta_a$$

Case 2.

The lines have *different phase* constants and *equal attenuation* constants.

κ is a real number

W changes asymptotically from

$$W_0 = 0 \text{ to}$$

$$W_1 = 0.172 \text{ and to}$$

$$W_\infty = 1$$

Case 3.

The *phase* and *attenuation* constants *differ by equal amounts.* As sbown below, in section 4, this case applies to the coupling between the TE_{01} and TM_{11} modes in curved circular wave guides with finite conductivity.

κ^2 is an imaginary number.

W changes asymptotically from

$$W_0 = 0 \text{ to}$$

$$W_1 = 0.217 \text{ and}$$

$$W_\infty = 1$$

For $\kappa \gg 1$ all three cases approach the limit

$$\beta_\infty \doteq \frac{\beta_1 + \beta_2}{2}\left(1 \mp \frac{k}{2}\right) \qquad 1.2\text{-}10$$

$$\alpha_\infty \doteq \frac{\alpha_1 + \alpha_2}{2}\left(1 \mp \frac{k}{2}\right) \doteq \frac{\alpha_1 + \alpha_2}{2}$$

2. Derivation of Field Equations

Consider a straight circular cylinder with an inside radius such as shown on Fig. 1 A. Let the radial coordinate equal r, the azimuthal coordinate equal φ and the longitudinal coordinate equal z. Let the dielectric losses inside the cylinder be negligible.

The field equations inside the cylinder are[5]

$$\frac{\partial E_z}{r\partial\varphi} - \frac{\partial E_\varphi}{\partial z} = -i\omega\mu\, H_r$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -j\omega\mu r\, H_\varphi$$

$$\frac{\partial(rE_\varphi)}{\partial r} - \frac{\partial E_r}{\partial\varphi} = -j\omega\mu r\, H_z$$

and

$$\frac{\partial H_z}{r\partial\varphi} - \frac{\partial H_\varphi}{\partial z} = j\omega\epsilon\, E_r$$

$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = j\omega\epsilon\, E_\varphi$$

$$\frac{\partial(rH_\varphi)}{\partial r} - \frac{\partial Hr}{\partial\varphi} = j\omega\epsilon r\, E_z$$

[5] See Ref. 4, pg. 94 of the book.

The natural transmission modes which satisfy these equations have the form

$$E = f_n(r)\,\epsilon^{jn(\varphi+\varphi_0)} \cdot \epsilon^{j\omega t - \Gamma z} \qquad 2\text{-}0$$

Each of these modes conforms to the same equations as a wave traveling in a transmission line with an impedance and phase velocity dependent upon the mode. In a straight cylinder with perfectly conducting walls, there exists no coupling between the different modes so that any and all can exist without interacting. If the conductivity of the walls in a straight circular cylinder is finite, it produces a resistive coupling between modes of equal azimuthal index (n in equation 2-0). In copper tubing and at the frequencies now obtainable ($\omega < 10^{12}$) this coupling effect is negligible.

A stronger coupling may be caused by deviations of the wave guide from the shape of a straight circular cylinder. The deformation considered in the present analysis consists in a circular bend of the axis, as shown schematically on Fig. 1b.

In such a circular bend the longitudinal coordinate is replaced by the product of the bending radius R by the bending angle θ:

$$z = R\theta$$

This transforms the first two component equations of curl E into

$$\frac{\partial(RE_\theta)}{Rr\partial\varphi} - \frac{\partial E_\varphi}{R\partial\theta} = -j\omega\mu\, H_r$$

$$\frac{\partial E_r}{R\partial\theta} - \frac{\partial(RE_\theta)}{R\partial r} = -j\omega\mu\, H_\varphi$$

The variable R can be eliminated by the relation

$$R = R_0 - r\cos\varphi$$

where R_0 is the bending radius of the cylinder axis. The coordinate θ can be replaced by a longitudinal coordinate s, measured along the cylinder axis. Hence,

$$s = \theta R_0$$

The progressive modes which we investigate have the approximate form

$$E = f_n(r)\,\epsilon^{jn(\varphi+\varphi_0)}\,\epsilon^{j\omega t - \Gamma s}$$

Hence

$$\frac{\partial}{\partial\theta} = R_0\frac{\partial}{\partial s} = -\; R_0\Gamma$$

$$\frac{\partial}{\partial\varphi} = jn \quad \text{for all field components.}$$

$\frac{\partial}{\partial r}$ may be expressed by a prime:

$$\frac{\partial F}{\partial r} = F'$$

Thus the equations with curl E become

$$\frac{jnE_z}{r} + \frac{E_z \sin\varphi}{R_0 - r\cos\varphi} + \frac{R_0 \Gamma E_\varphi}{R_0 - r\cos\varphi} = -j\omega\mu H_r$$

$$\frac{-R_0\Gamma E_r}{R_0 - r\cos\varphi} - E_z' + \frac{E_z \cos\varphi}{R_0 - r\cos\varphi} = -j\omega\mu H_\varphi$$

$$E_\varphi + rE_\varphi' = jnE_r = -j\omega\mu r H_z$$

For gradual bends

$$R_0 \gg a > r$$

One may therefore approximate

$$\frac{R_0}{R_0 - r\cos\varphi} \doteq 1 + \frac{r}{R_0}\cos\varphi$$

It is convenient to introduce the symbol

$$c = \frac{a}{R_0}$$

which is proportional to the coupling coefficient. All powers of c greater than the first will be neglected. One can now write the approximate field equations in the curved cylinder:

$$\frac{jnE_z}{r} + \Gamma E_\varphi + \frac{c}{a} E_z \sin\varphi + c\Gamma \frac{r}{a} E_\varphi \cos\varphi = -j\omega\mu H_r \qquad 2\text{-}1$$

$$-\Gamma E_r - E_z' - c\Gamma \frac{r}{a} E_r \cos\varphi + \frac{c}{a} E_z \cos\varphi = -j\omega\mu H_\varphi \qquad 2\text{-}2$$

$$E_\varphi + rE_\varphi' - jnE_r = -j\omega\mu r H_z \qquad 2\text{-}3$$

$$\frac{jnH_z}{r} + \Gamma H_\varphi + \frac{c}{a} H_z \sin\varphi + c\Gamma \frac{r}{a} H_\varphi \cos\varphi = j\omega\epsilon E_r \qquad 2\text{-}4$$

$$-\Gamma H_r - H_z' - c\Gamma \frac{r}{a} H_r \cos\varphi + \frac{c}{a} H_z \cos\varphi = j\omega\epsilon E_\varphi \qquad 2\text{-}5$$

$$H_\varphi + rH_\varphi' - jnH_r = j\omega\epsilon r E_z \qquad 2\text{-}6$$

The coupling terms all contain the factor $\cos\varphi$ or $\sin\varphi$. This means that every transmission mode is coupled only to modes with an azimuth index differing from its own by ± 1.

3. Characteristic Equations of TE_{01} and TM_{11} Modes

The mode which one desires to propagate through the wave guide is the TE_{01} mode. In a straight wave guide with perfectly conducting walls it is characterized by the following equations:

$$n = 0 \tag{3-1}$$

$$E_{r1} = E_{z1} = H_{\varphi 1} = 0 \tag{3-2}$$

$$E_{\varphi 1} = E_1 \epsilon^{j\omega t - \Gamma_1 z} J_1(y) = e_1 J_1(y) \tag{3-3}$$

$$H_{r1} = -\frac{e_1 \Gamma_1}{j\beta_0 \eta} J_1(y) \tag{3-4}$$

$$H_{z1} = -\frac{e_1 \chi}{j\beta_0 \eta} J_0(y) \tag{3-5}$$

In these equations

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = 377 \text{ ohm (intrinsic free space resistance)} \tag{3-6}$$

$$\beta_0 = \omega\sqrt{\epsilon\mu} = \frac{2\pi}{\lambda_0} \tag{3-7}$$

$$y = \chi r \tag{3-8}$$

$$\Gamma_1 = \sqrt{\chi^2 - \beta_0^2} \tag{3-9}$$

In a perfectly conducting wave guide

$$\chi_0 = \frac{3.832}{a} \tag{3-10}$$

$$\nu = \frac{\chi_0}{\beta_0} = \frac{.61\lambda_0}{a} \text{ (cutoff factor)} \tag{3-11}$$

$$\Gamma_1 = j\beta_0\sqrt{1 - \nu^2} = j\beta_1 \tag{3-12}$$

If the wave guide has the conductivity of g mho/m, its intrinsic high frequency impedance is

$$Z_i = (1 + j)R_i = (1 + j)34.4\sqrt{\frac{\mu}{g\lambda_0}}\,* \tag{3-13}$$

This changes χ to

$$_1\chi \doteq \chi_0 - \frac{(1 - j)R_i\nu}{\eta a} \text{ and} \tag{3-14}$$

$$\Gamma_1 \doteq j\beta_1 + \frac{\nu^2}{\sqrt{1 - \nu^2}}\,\frac{(1 + j)R_i}{a\eta}\,\dagger \tag{3-15}$$

* Ref. 4 pg. 83.
† Compare Ref. 4, pg. 390.

Due to the curvature of the guide, this desired mode is coupled to all modes which have the azimuthal index number 1.

However, for low curvatures, this coupling is very loose and only causes appreciable effects if it can act over a great length of wave guide without phase interference.

This means that the disturbing mode must have nearly the same phase velocity as the desired mode. It so happens that in a perfectly conducting circular cylinder there exists one mode, the TM_{11}, which has exactly the same velocity as the TE_{01}. Such a coincidence is called "degeneracy."

In the analysis of very gradual bends, only this TM_{11} mode need be considered. It is characterized in a straight guide by the following equations:

$$n = 1 \tag{3-16}$$

$$E_{\varphi 2} = E_2 \epsilon^{j\omega\tau - \Gamma_2 z} \cdot \frac{J_1(y)}{y} \cos(\varphi + \varphi_0) = e_2 \frac{J_1(y)}{y} \cos\varphi \tag{3-17}$$

The TM_{11} mode can be polarized in all directions. But since only the component directed toward

$$\varphi_0 = 0$$

is excited by the wave guide curvature, φ_0 has been omitted in the last term of eq. (3-17).

$$E_{r2} = e_2 \frac{dJ_1(y)}{dy} \sin\varphi = e_2 \dot{J}_1(y) \sin\varphi, \quad \text{where } \dot{J} = \frac{dJ}{dy}$$

$$E_{z2} = \frac{-\chi_2 e_2}{\Gamma_2} J_1(y) \sin\varphi$$

$$H_{\varphi 2} = \frac{e_2 j\beta_0}{\eta\Gamma_2} \dot{J}_1(y) \sin\varphi$$

$$H_{r2} = -\frac{e_2 j\beta_0}{\eta\Gamma_2} \frac{J_1(y)}{y} \cos\varphi$$

$$H_{z2} = 0$$

In a perfectly conducting wave guide the χ defined by eq. 3-8 is

$$\chi = \chi_2 = \chi_0 \text{ and}$$

$$\Gamma_2 = \Gamma_1 = j\beta_1$$

In a wave guide with an intrinsic impedance per 3-13,

$$\chi_2 \doteq \chi_0 - \frac{(1-j)R_i}{\eta a \nu} \quad \text{and} \tag{3-18}$$

$$\Gamma_2 \doteq j\beta_1 + \frac{(1+j)R_i\,\dagger}{\sqrt{1-\nu^2}\, a\eta} \tag{3-19}$$

From 3-15 and 3-19 one finds $\alpha_1 = \alpha_2 \nu^2$ 3–20

4. Interaction Between TE_{01} and TM_{11} Modes

Since the separate modes of propagation behave like traveling waves in transmission lines[2], their interaction can be derived from the coupling equations derived in Section 1 of this analysis.

The uncoupled propagation constants Γ_1 and Γ_2 are known (equations 3-15 and 3-19). In order to find the coupling discriminant one must derive the coupling coefficient from the field equations 2-1 to 2-6. The coupling coefficient is defined as

$$k = \frac{P_{12}}{\sqrt{P_1 P_2}} = \frac{P_{21}}{\sqrt{P_1 P_2}} \qquad \text{(eq. 1.2-4)}$$

In computing the coupling coefficient one may neglect the small attenuation constant. The energy stored by the TE_{01} wave per unit length is

$$P_1 = \int_0^a \frac{E_1^2}{\eta} \cdot 2\pi r \, dr = \int_0^a H_1^2 \eta \cdot 2\pi r \, dr \qquad 4\text{-}1$$

This expression is not affected by the cutoff factor ν because one may consider the field inside the guide as composed of slanting plane waves with the electric field strength E_1. The energy stored by the TM_{11} wave per unit length is

$$P_2 = \int_0^{2\pi} \int_0^a \frac{E_2^2}{\eta} r \, d\varphi \, dr = \int_0^{2\pi} \int_0^a H_2^2 \eta r \, d\varphi \, dr$$

The interchanged energy:

$$P_{21} = \int_0^{2\pi} \int_0^a \frac{E_{21} E_1}{\eta} \, d\varphi \, dr + \int_0^{2\pi} \int_0^a H_{21} H_1 \eta \, d\varphi \, dr$$

Combining equation (4-1) with 3-3, 3-8 and 3-10

$$P_1 = \frac{2\pi a^2 e_1^2}{3.832^2 \eta} \int_0^{3.832} y \delta_1^2 J_1(y) \, dy$$

From reference 1, page 146 of the book

$$\int_0^{3.832} y \delta_1^2 J_1(y) \, dy = \frac{3.832^2}{2} J_0(3.832)$$

Hence

$$P_1 = \frac{0.51 e_1^2 a^2}{\eta} \qquad 4\text{-}2$$

In a similar manner one finds

$$P_2 = \frac{0.51 e_2^2 a^2}{2(1 - \nu^2)\eta} \qquad 4\text{-}3$$

[2] Loc. cit.

and

$$P_{12} = P_{21} = \frac{0.51 ce_1 e_2 a^2}{3.832\eta} \qquad 4\text{-}4$$

Substituting the values of 4-2, 4-3 and 4-4 into 1.2-4 one finds for the coupling coefficient between the two groups of plane waves traveling at a slant to the wave guide axis

$$k_s = \frac{c\sqrt{2}\sqrt{1 - \nu^2}}{3.83} = \frac{0.369a}{R_0}\sqrt{1 - \nu^2}$$

The coupling coefficient k between the TE_{01} and TM_{11} modes which are the resultants of their slant wave groups is greater than k_s according to the following reasoning:

From 1.2-10

$$\beta \doteq \beta_0(1 \pm 0.5\, k_s)$$

This makes the cutoff factors of the coupled modes

$$\nu = \frac{x}{\beta} = \frac{\nu_0}{1 \pm 0.5k_s}$$

and the coupled propagation constants

$$\Gamma = \beta\sqrt{1 - \nu^2} = \beta_0\sqrt{(1 \pm 0.5k_s)^2 - \nu_0^2}$$

For $k_s \ll 1$

$$\Gamma \doteq \beta_0\sqrt{1 - \nu_0^2}\left(1 \pm \frac{0.5k_s}{1 - \nu_0^2}\right) = \Gamma_0(1 - 0.5k)$$

Hence, in view of eq. 1.2-10, the effective coupling coefficient of the wave guide modes is

$$k = \frac{k_s}{1 - \nu^2} = \frac{0.369a}{R_0\sqrt{1 - \nu^2}} \qquad 4\text{-}5$$

From equations 3-15 and 3-19

$$\Gamma_1 - \Gamma_2 = -\frac{1 + j}{a\eta} R_i\sqrt{1 - \nu^2} \qquad 4\text{-}6$$

$$\frac{\sqrt{\Gamma_1\Gamma_2}}{\Gamma_1 - \Gamma_2} \doteq \frac{j\beta_0}{\Gamma_1 - \Gamma_2} \doteq \frac{-2\pi\eta a}{(1 - j)R_i\lambda_0\sqrt{1 - \nu^2}} \qquad 4\text{-}7$$

From 1.2-3, 4-5 and 4-7 one obtains the coupling discriminant

$$\kappa = \frac{-\,0.369\cdot 2\pi\cdot 377a^2}{R_0R_i\lambda_0(1 - j)(1 - \nu^2)}$$

Since

$$R_i = 0.00452\,\lambda_0^{-0.5}\cdot\rho_r\,^* \qquad 4\text{-}8$$

* Loc. cit.

with the relative high frequency resistivity

$$\rho_r = \sqrt{\frac{\rho}{\rho_{copper}}}$$

$$\kappa = -\frac{9.65 \cdot 10^4(1 + j)a^2\lambda^{-0.5}}{R_0\,\rho_r(1 - \nu^2)}$$

Its absolute value

$$|\kappa| = \frac{1.366 \cdot 10^5 a^2 \lambda^{-0.5}}{R_0\,\rho_r(1 - \nu^2)}$$

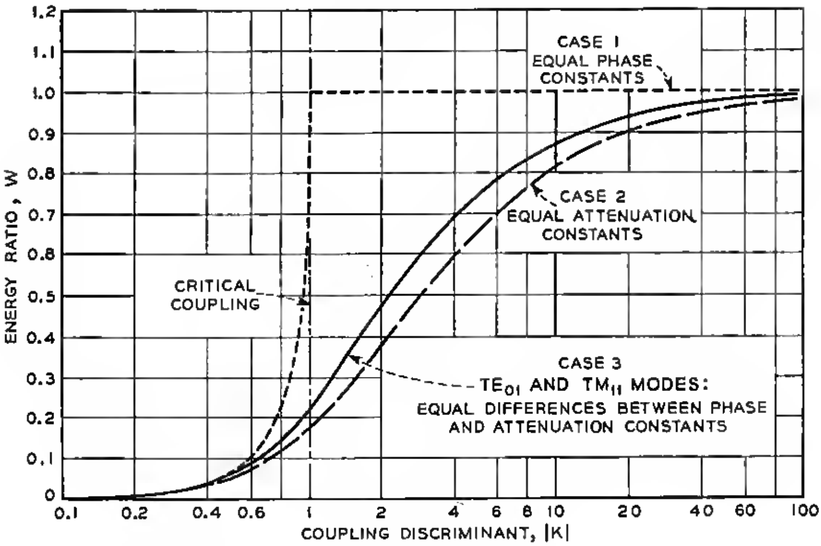


Fig. 3

As shown in equation 4-6, tbe differences between the propagation constants of the TM_{11} and TE_{01} waves are proportional to the intrinsic skin impedance R_i which contains the factor $(1 + j)$. This means that the phase and attenuation constants of the two waves differ by equal amounts in accordance with "*Case 3*" of Section 1. For this case the power ratio per 1.2-20 may be written

$$W = \left| \frac{\sqrt{i + j\,|\kappa|^2} - 1}{\sqrt{1 + j\,|\kappa|^2} + 1} \right|$$

The numerical value of this function is plotted on Fig. 3. It can be computed conveniently by means of the following auxiliary parameters:

$$\sqrt{1 + j\,|\kappa|^2} = p + jq = \cosh x + j \sinh x \qquad 4\text{-}9$$

$$p = \sqrt{0.5 + 0.5\sqrt{1 + |\kappa|^4}} = \cosh x$$
$$q = \sqrt{-0.5 + 0.5\sqrt{1 + |\kappa|^4}} = \sinh x$$
$$W = \frac{p - 1}{p} \tanh \frac{x}{2}$$

In analogy to *Case 1* of Section 3, the condition of $|\kappa| = 1$ may be called *critical coupling*. It occurs at the *critical radius of curvature*

$$R_{0\,\mathrm{cr}} = \frac{1.366 \cdot 10^5\, a^2 \lambda^{-0.5}}{\rho_r(1 - v^2)} \text{ meters} \tag{4-10}$$

For subcritical coupling ($R_0 \gg R_{11}$) W approaches

$$W_{\mathrm{subcr.}} \doteq \left| \frac{\kappa^2}{4} \right| \longrightarrow 0$$

For supercritical coupling ($R_0 \ll R_{\mathrm{cr}}$),

$$W_{\mathrm{supercr.}} \doteq 1 - \frac{\sqrt{2}}{|\kappa|} \longrightarrow 1$$

From the above results, it is possible to predict the behavior of waves originating either as TE_{01} or as TM_{11} modes, in any given wave guide configuration. This is done for some typical cases in the following sections.

5. Propagation in Long Waveguides with Constant Curvature

It has been shown in Section 3 that for each curvature there exist two modes of propagation.

In one,

$$W_a = \frac{P_{\mathrm{TM11}}}{P_{\mathrm{TE01}}} < 1 \tag{5-1}$$

and the attenuation

$$\alpha_a < \frac{\alpha_{\mathrm{TE}} + \alpha_{\mathrm{TM}}}{2} \tag{5-2}$$

In the other,

$$W_b = \frac{1}{W_a} > 1 \tag{5-3}$$

and the attenuation

$$\alpha_b > \frac{\alpha_{\mathrm{TE}} + {}_{\mathrm{TM}}}{2} \tag{5-4}$$

In a long wave guide the "b" mode will die down due to its greater attenuation, no matter how much of it was initially present, so that one need only consider the "a" mode.

This mode has a phase velocity slightly smaller than that of the uncoupled TE_{01} wave and an attenuation nearer to that of the TE_{01} than the TM_{11} wave.

The magnitude of the critical radius is illustrated hy the two examples of Tahle I.

TABLE I
CHARACTERISTIC VALUES

Parameter	Symbol	Equation	Example 1	Example 2
Wave guide radius	a		.05 m	.05 m
Free space wave length	λ_0		.03 m	.01 m
Cutoff ratio	ν	3-11	.366	.122
Attenuation constant	$\alpha_{TE(cu)}$	3-15, 4-17	2.04×10^{-4} neper/m	3.58×10^{-5} neper/m
Attenuation constant	$\alpha_{TM(cu)}$	3-19	1.53×10^{-3} neper/m	2.41×10^{-3} neper/m
Critical Radius	R_{crit}	4-10	2.12 km.	3.44 km.

TABLE II
RELATIVE ATTENUATION VERSUS RADIUS OF CURVATURE

General formulae				Example 1		Example 2	
κ	R_0/R_{cr}	W	α/α_0	$R_0 km$	α/α_0	$R_0 km$	α/α_0
0	∞	0	1	∞	1.00	∞	1.00
0.1	10	0.0025	$1 + 0.0025\ (\nu^{-2} - 1)$	19.7	1.02	34.15	1.17
0.2	5	0.01	$1 + 0.01$ "	9.85	1.06	17.08	1.66
0.5	2	0.06	$1 + 0.057$ "	3.84	1.38	6.83	4.45
1	1	0.22	$1 + 0.18$ "	1.97	2.16	3.42	12.94
2	0.5	0.48	$1 + 0.32$ "	0.98	3.11	1.71	22.2
5	0.2	0.75	$1 + 0.43$ "	0.38	3.83	0.68	26.6
10	0.1	0.87	$1 + 0.46$ "	0.20	4.05	0.34	27.9
∞	0	1.00	$1 + 0.50$ "	0	4.30	0	34.1

The increase of attenuation in long wave guides with uniform curvature is shown on Tahle II, with numerical values for the same examples as in Tahle I.

6. Propagation in a Uniformly Curved Section of Wave Guide Following a Long Straight Section. (Fig. 4)

No matter what mixture of modes may prevail at the heginning of the wave guide, all modes except the TE_{01} die down in the long straight section due to their higher attenuation, so that the wave form at the heginning of the curved section is pure TE_{01}.

Since it has heen shown in Sections 1 and 4 that each of the two possihle

modes of propagation in a curved wave guide consists of both TE_{01} and TM_{11} waves, it follows that both modes must be superimposed in such a manner that at the transition point the TM components cancel each other by interference.

Let the relative amplitudes of the two TE components equal a and b; then the corresponding amplitudes of the TM modes are aA_a and bA_b

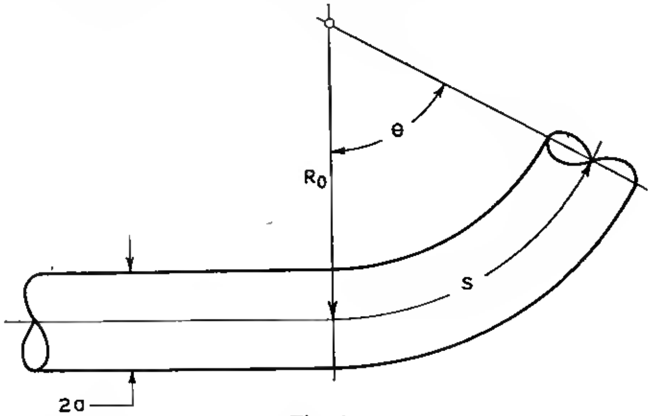


Fig. 4

respectively, where A_a and A_b are the normalized voltage ratios per 1.2-5. At the beginning of the curved section

$$a + b = 1 \qquad \text{(TE amplitude)}$$

$$aA_a + bA_b = aA_a - b/A_a = 0 \qquad \text{(TM amplitude)}$$

Hence

$$a = \frac{1}{1 + A_a^2}$$

$$b = \frac{A_a^2}{1 + A_a^2}$$

$$\left|\frac{b}{a}\right| = \left|A_a^2\right| = W$$

The two waves have different phase velocities and therefore interfere with each other. According to 1.2-9, the difference between the phase constants is

$$\beta_b - \beta_a = (\beta_2 - \beta_1)\,\frac{1 + W}{1 - W}$$

By means of equations 4-9, this can be transformed into

$$\beta_b - \beta_a = (\beta_2 - \beta_1)(p + q) \qquad 6\text{-}1$$

The length of one complete interference cycle is

$$s_{2\pi} = \frac{2\pi}{(p + q)(\beta_2 - \beta_1)}$$

If both components had equal attenuation, the beats superimposed on the decaying envelope of the TE_{01} wave would correspond to an amplitude ratio

$$\frac{e_0 \max}{e_0 \min} = \frac{1 + W}{1 - W} = p + q$$

However, during the progress of the mixed wave through the curved section, the intensity of the fluctuations is reduced by the greater attenuation constant of the faster and weaker "b" component. In one complete interference cycle, the differential attenuation reduces the weaker component to

$$\frac{A_{2\pi}}{A_0} = \epsilon^{-(\alpha_2 - \alpha_1)s_{2\pi}} = \epsilon^{-(2\pi/p+q)}$$

Approximation for Weak Coupling

For $|\kappa| \ll 1$

$\beta_b - \beta_a \doteq (\beta_2 - \beta_1)(1 + 0.5\,|\kappa|^2)$

From 3-15 and 3-19

$$\beta_b - \beta_a \doteq \frac{R_i}{a\eta}\sqrt{1 - \nu^2}$$

$$\doteq \frac{1.2 \times 10^{-5}}{a}\sqrt{\frac{(1 - \nu^2)g_{cu}}{\lambda_0 g}}\,(1 + 0.5\,|\kappa|^2) \text{ radian}/m$$

Fnr intermediate coupling, 6-1 may be transformed into

$$\beta_b - \beta_a = k\beta_1 f(\kappa)$$

$$\text{with } f_{(\kappa)} = \frac{p + q}{\sqrt{2}\,|\kappa|} = \frac{\sqrt{1 + \sqrt{1 + |\kappa|^4}} + \sqrt{-1 + \sqrt{1 + |\kappa|^4}}}{2\,|\kappa|}$$

Approximation for Strong Coupling

For $|\kappa| \gg 1$

$$f_{(\kappa)} \doteq 1 + 0.125\,|\kappa|^{-4}\;\beta_b - \beta_a \doteq k\beta_1(1 + 0.125\,|\kappa|^{-4})$$

Substituting the value of k from 4-13 and transforming,

$$\beta_b - \beta_a = \frac{2.32a}{\lambda_g R_0} f_{(\kappa)} = \frac{2.32a}{\lambda_0 R_0}(1 + 0.125\,|\kappa|^{-4}) \qquad 6\text{-}2$$

The phase difference between the two components is

$$\Psi_{b-a} = \frac{2.32as}{\lambda_0 R_0} f_{(\kappa)} = \frac{2.32a\theta}{\lambda_0} f_{(\kappa)} = M\theta \qquad 6\text{-}3$$

where θ is the bending angle of the wave guide. The power carried by the TE_{01} wave is

$$P_{TE} = \cos^2 \frac{M\theta}{2}. \qquad 6\text{-}4$$

Minima of TE_{01} occur when this phase difference is an odd multiple of π. Hence, the bending angles producing minima of TE_{01} amplitudes are:

$$\theta_{\min} \doteq (2n + 1) \cdot \frac{1.36\lambda_0}{a(1 + 0.125\,|\,\kappa\,|^{-4})} \doteq \frac{(2n + 1)2.22\nu}{f_{(\kappa)}} \qquad 6\text{-}5$$

The initial fluctuation ratio approaches

$$\frac{e_{1\,\max}}{e_{1\,\min}} = p + q \doteq \sqrt{2}\,|\,\kappa\,|$$

which is a large value tending to infinity.

The relative attenuation of the slightly weaker component during one beat cycle is

$$\frac{A_{2\pi}}{A_0} = \epsilon^{-2\pi/p+q} = \epsilon^{-\sqrt{2}\pi/|\kappa|} \doteq 1 - \frac{4.44}{|\,\kappa\,|}$$

which is a small reduction tending to zero. Hence, the fluctuations persist through a large number of beats. The power is transformed back and forth between the TE_{01} and the TM_{11} modes.

In Section 5, it was shown that in a long, uniformly curved wave guide the attenuation is intermediate between that of the TE_{01} and TM_{11} modes. But from equations 1.2-7 and 8 it follows that the two modes contribute to the attenuation in proportion to their relative power flow. Since at the beginning of the bend the power of the TM_{11} component is zero, it is to be expected that the *initial* rate of attenuation equals that of the TE_{01} wave alone. This is proved by differentiating with regard to s. One finds for all values of κ that

$$\frac{d}{ds}\left| a\epsilon^{-\Gamma_a s} + b\epsilon^{-\Gamma_b s} \right|_{s=0} = -\alpha_1$$

Discussion of Results

Equation 6-2 corresponds directly to an equation derived by S. O. Rice and, after allowing for the different choice of variables, to M. Jouguet's equation (75)[6]. It differs from the results of these earlier calculations by the factor $f(\kappa) \doteq 1 + 0.125\,|\,\kappa\,|^{-4}$ which is a reminder that the simplified form of the equations given by the earlier authors is an extrapolation to infinite conductivity or infinite curvature of the wave guide.

[6] Reference 3, pg. 150 of *Cables and Transmission*, July 1947.

From equations 6-3 and 6-4 it is seen that the TE_{01} wave is recovered by bends which are an even multiple of $\theta_{\min}$. But such bends are efficient transmitters of TE_{01} waves only over a narrow frequency range since θ_{min} varies with frequency.

If the circular hend is followed hy a long straight section, the TE_{01} and TM_{11} components existing at the end of the hend are carried over into the straight section, hut the TM_{11} component dies down due to its greater attenuation and constitutes a total loss.

Numerical examples for first extinction angle.

Using the same dimensions as in Tahle I of Section 5, one finds from eq. 6-5 for:

$$\text{Example 1: } \theta_{min} \doteq 0.816 \text{ Radians} \doteq 46.8°$$

$$\text{Example 2: } \theta_{min} \doteq 0.272 \text{ Radians} \doteq 15.6°$$

7. Serpentine Bends

Sections 5 and 6 dealt with bends continued with uniform curvature over large angles. The present section considers the small random deviations from a straight course which are unavoidahle in field installations.

Actual deviations are expected to he random hoth with regard to maximum deflection angle and to curvature; they are likely to approximate a sinusoidal shape. For purposes of computation, the following analysis assumes as a first case *circular S-bends* which consist of alternate regions of equal lengths and equal hut opposite curvatures. An exaggerated schematic of such S-bends is shown on Fig. 5A.

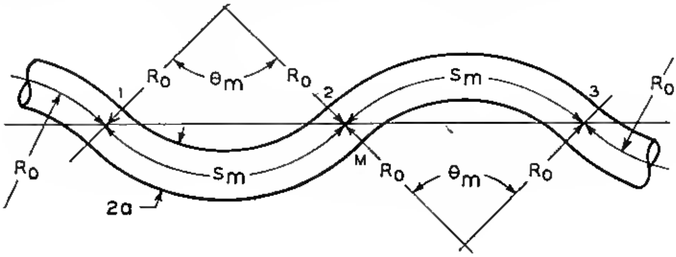
Each circular hend tends to produce a single mode with an attenuation per equation 1.2-7. However, the discontinuous reversals of curvature at the inflexion points produce mixed modes, and the initial part of each region reduces the amplitude of the TM components produced in the previous region.

Each region may he treated as a discrete 4-terminal section of a transmission network. Regardless of the wave composition at the input terminal, differential attenuation will estahlish in a long serpentine wave guide a steady state condition. In this steady state each region produces equal attenuation. This attenuation per region and the resulting average attenuation constant will now be derived.

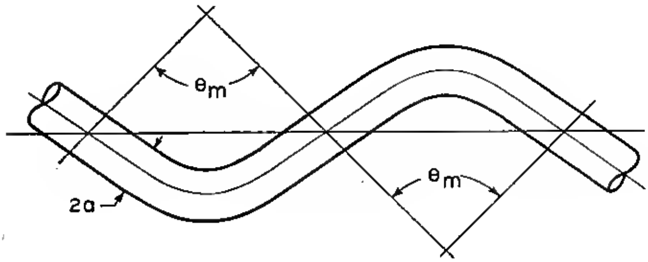
The TE_{01} and TM_{11} waves each consist of "a" and "b" components with separate amplitude ratios and propagation constants, as derived in Sections 1 and 4. In the first region (hetween points 1 and 2 of Fig. 5A) R_0 is taken as positive, and

$$A = \sqrt{1 + \kappa^{-2}} - \kappa^{-1} \tag{1.2-6}$$

In the second region, between points 2 and 3, the polarity of R_0, and consequently of κ and the ratio of TM to TE amplitudes, are reversed.



a - CIRCULAR 'S' BENDS



b - SINUSOIDAL BENDS

Fig. 5

Except for this change of polarity the amplitude ratios at points 1 and 2 are equal. Introducing the symbols

$$g = \epsilon^{-\Gamma s m}$$
$$g_m = \tfrac{1}{2}(g_a + g_b)$$
$$g_d = \tfrac{1}{2}(g_b - g_a)$$

one can tabulate

point	e_{TE}	e_{TM}	
1	$a + b$	$aA - \dfrac{b}{A}$	7-1
2	$g_m\left(\dfrac{a}{g_d} + b g_d\right)$	$g_m\left(\dfrac{aA}{g_d} - \dfrac{b}{A}\, g_d\right)$	7-2

Calling $\dfrac{b}{a} = y$, one finds

$$\frac{1 + y g_d^2}{1 + y} = \frac{-A^2 + y g_d^2}{A^2 - y} = \frac{g_d}{g_m} \cdot g_{\text{average}} = \frac{g_{\text{average}}}{g_a}$$

$$\Gamma_{\text{average}} = \Gamma_a + \frac{1}{s_m} \log \frac{1 + y}{1 + y g_d^2} \text{ with}$$

$$y = \frac{A}{2} \kappa^{-1}(1 + g_d^{-2}) + A \sqrt{g_d^{-2} + \frac{\kappa^{-2}}{4}(1 + g_d^{-2})^2}$$

This formal solution is hard to evaluate. It can be greatly simplified for the subcritical and supercritical cases.

1. *Subcritical Curvature* $|\kappa| \ll 1$

$$y \doteq \frac{\kappa^2}{2 + 2_d^2}$$

$$\Gamma_{\text{average}} = \Gamma_a \left(1 + \frac{\kappa^2}{2} \frac{1 - g_d^2}{1 + g_d^2}\right) \doteq \Gamma_a \doteq \Gamma_1$$

For very low curvatures, the average attenuation approaches that of the "a" mode, and this in turn approaches that of the TE_{01} wave.

2. *Supercritical Curvature.* $|\kappa| \gg 1$

The differential attenuation constant is small compared to the differential phase constant.

$|y| \doteq |A| \doteq 1$

Substituting these values into 7-1 and 2, one finds

$y \doteq \epsilon^{-0.5j\theta_m}$

Expressed as a function of θ:

$y_\theta = \cos \psi + j \sin \psi$ with

$\psi = M(\theta - 0.5\ \theta_m)$ 7-3

M bas the value per eq. 6-18.

The power ratio of the combined TM_{11} and TE_{01} waves is

$W_\theta \doteq \tan^2 \psi/2$

In view of equation 1.2-7 the instantaneous rate of energy loss is

$$\alpha_\theta = \alpha_1 \cos^2 \frac{\psi}{2} + \alpha_2 \sin^2 \frac{\psi}{2} = \alpha_1 + (\alpha_2 - \alpha_1) \sin^2 \frac{\psi}{2} \qquad \text{7-4}$$

$$\alpha_{\text{average}} = \frac{1}{s_m} \int_0^{s_m} \alpha_s\, d_s = \frac{1}{\theta_m} \int_0^{\theta_m} \alpha_\theta\, d\theta$$

$$\alpha_{\text{average}} \doteq \alpha_1 + (\alpha_2 - \alpha_1) \left(\frac{1}{2} - \frac{\sin M\theta_m}{2M\theta_m}\right)$$

In view of 3-20

$$\alpha_{\text{average}} = \alpha_1 \left[\frac{\nu^{-2} + 1}{2} - \frac{\nu^{-2} - 1}{2} \frac{\sin M\theta_m}{M\theta_m}\right]$$

For small deflection angles, $M\theta_m \ll 2$

$$\alpha_{\text{average}} \doteq \alpha_1\left[1 + \frac{\nu^{-2} - 1}{6} M^2\theta_m^2\right]$$

If a $p\%$ increase in attenuation is the tolerance limit

$$\theta_m \leqq \frac{1}{10M}\sqrt{\frac{6p}{\nu^{-2} - 1}}.$$ Substituting the value of M from 6-3,

$$\theta_m = \frac{0.105\sqrt{p}\,\nu\lambda_0}{a\sqrt{1 - \nu^2}}.$$

The maximum deflection equals

$\Delta_\theta = 0.5\,\theta_m$. Hence, in view of 3-11

$$\Delta_\theta \leqq \frac{0.032\sqrt{p}\lambda_0^2}{a^2\sqrt{1 - \nu^2}} \text{ radians} = \frac{1.84\lambda_0^2}{a^2}\sqrt{\frac{p}{1 - \nu^2}} \text{ degrees}$$

3. *Sinusoidal Bends with Predominantly Supercritical Curvature.*

Sinusoidal bends cannot be supercritically curved over their entire length because at the inflection points the curvature drops to zero. For sufficiently short bends, however, no great error is caused by treating the entire length as supercritical. In that case, equations 7-3 and 7-4 remain valid. θ takes the new value

$$\theta = \frac{\theta_m}{2} + \frac{\theta_m}{2}\sin\frac{\pi(s - s_m)}{2s_m}$$

Hence

$$\psi = \frac{M\theta_m}{2}\sin\frac{\pi(s - s_m)}{2s_m}$$

$$\alpha_{\text{average}} = \alpha_1 + \frac{\alpha_2 - \alpha_1}{s_m}\int_0^{s_m}\sin^2\left[\frac{M\theta_m}{2}\sin\frac{\pi(s - s_m)}{2s_m}\right]ds$$

For small deflection angles, $M\theta_m \ll 2$

$$\alpha_{\text{average}} = \alpha_1 + \frac{\alpha_2 - \alpha_1}{s_m}\frac{M^2\theta_m^2}{4}\int_0^{s_m}\sin^2\frac{\pi(s - s_m)}{2s_m}\,ds$$

$$= \alpha_1 + (\alpha_2 - \alpha_1)\frac{M^2\theta_m^2}{4} = \alpha_1\left[1 + \frac{\nu^{-2} - 1}{4}M^2\theta_m^2\right]$$

$$\Delta_\theta = \frac{0.026\lambda_0^2\sqrt{p}}{a^2\sqrt{1 - \nu^2}} \text{ radians} = \frac{1.49\lambda_0^2}{a^2}\sqrt{\frac{p}{1 - \nu^2}} \text{ degrees}$$

The tolerance limit for sinusoidal deflections is 20% smaller than for circular S bends.

The effect of supercritical but shallow circular and sinusoidal S bends is illustrated by the following *numerical examples.*

TABLE III
INCREASE OF ATTENUATION IN S BENDS

Attenuation Increase p%	Maximum deflection $\Delta\Theta$ (in degrees)			
	Example 1 (ν = .366)		Example 2 (ν = .122)	
	Circular	Sinusoidal	Circular	Sinusoidal
10	2.25	1.82	0.23	0.19
20	3.18	2.58	0.33	0.27
30	3.89	3.15	0.41	0.33
40	4.50	3.64	0.47	0.38
50	5.03	4.07	0.52	0.42

8. Helical Bends and Random Two-Dimensional Deviations

A helical bend may be treated as a bend which has a constant absolute magnitude, but a changing direction of curvature. As indicated in eq. 3-17, the TM_{11} wave can be polarized in all directions. At any differential element of wave guide length, the TM_{11} component polarized in the local bending plane is coupled to the TE_{01} wave; the TM_{11} component polarized at right angles is not coupled and persists unchanged. By requiring that the absolute magnitude of the TM_{11}/TE_{11} amplitude ratio remain constant, a steady state solution can be found.

Shallow helical bends of small curvatures may be treated as the superposition of two sinusoidal bends offset by 90° in the longitudinal direction and in the bending plane. The increases in attenuation due to these two sinusoidal bends are computed from eq. 7-5 and added.

It is believed that random deviations from a straight course approach sinusoidal shape more closely than circular shape, hence equation 7-5 may be used to establish a tolerance limit for such random deviations. For quantitative results the statistical distribution of the squared deviation maxima must be taken into consideration.

9. Optima of Wave Guide Radius, Signal Wave Length and Attenuation as a Function of Angular Deviation

In a straight wave guide the attenuation decreases with wave guide radius and signal frequency. However, the deterioration due to wave guide curvature increases with wave guide radius and frequency. Hence, for a given tolerance limit to angular deviation from the straight course there exists an optimum radius for each wave length and an optimum wave length for each radius. This will be shown for the case of uniform sinusoidal bends, under the simplifying assumption that the cutoff radio $\nu \ll 1$.

Solving 7-5 for p one obtains

$$p \doteq 0.45\Delta^2 a^4 \lambda^{-4} \qquad 9\text{-}1$$

where p si the percentage increase in attenuation, and Δ the deviation angle in degrees.

Hence the average attenuation

$$\alpha_\Delta = \alpha(1 + 0.01p) \qquad 9\text{-}2$$

From 3–15 and 3–11

$$\alpha \doteq \frac{\nu^2 R_i}{a\eta} \doteq 10^{-3} R_i \lambda^2 a^{-3} \qquad 9\text{-}3$$

Introducing the R_i value from 4-8

$$\alpha \doteq 4.5\ 10^{-5} \rho_1 \lambda^{1.5} a^{-3} \qquad 9\text{-}4$$

where ρ is the high-frequency resistance of the wave guide relative to copper. From 9-1, 2 and 4

$$\alpha_\Delta = 4.5\ 10^{-6} \rho \lambda^{1.5} a^{-3}(1 + q\lambda^{-4} a^4) \qquad 9\text{-}5$$

with

$$q = 4.5\ 10^{-3}\Delta^2$$

The attenuation reaches a minimum when

$$f(\lambda, a) = \lambda^{1.5} a^{-3} + q\lambda^{-2.5} a = \text{minimum}$$

Case 1. λ is given

$$\delta f/\delta a = -3\lambda^{1.5} a^{-4} + q\lambda^{-2.5} = 0$$

$$a_{opt} = 1.32\lambda q^{-0.25} = 5.2\lambda\Delta^{-0.5}$$

From 9-5

$$\alpha_{\Delta opt} = 4\alpha = 1.29\ 10^{-7} \rho \lambda^{-1.5} \Delta^{1.5}$$

Case 2. a is given

$$\delta f/\delta\lambda = 1.5\lambda^{0.5} a^{-3} - 2.5q\lambda^{-3.5} a = 0$$

$$\lambda_{opt} = 1.14 a q^{0.25} = 0.294 a \Delta^{0.5}$$

From 9-5

$$\alpha_{\Delta opt} = 1.6\alpha = 1.15\ 10^{-6} \rho a^{-1.5} \Delta^{0.75}$$

Numerical Example

Let $\Delta = 0.42°$
$a = 0.05\ m$
$\lambda = 0.01\ m$

From Table III

$$\alpha_\Delta = 1.50\,\alpha = 5.4\ 10^{-5}\rho \text{ neper}/m$$

Case 1: λ fixed at 0.01 m

$$a_{opt} = 0.08\ m$$

$$\alpha_{opt} = 2.76\ 10^{-5}\rho \text{ neper/meter}$$

Case 2: a fixed at 0.05 m

$$\lambda_{opt} = 0.0097\ m$$

$$\alpha_{opt} = 5.36\ 10^{-5}\rho \text{ neper}/m$$

Assuming sinusoidal bends with a 0.42° maximum deviation, the attenuation of centimeter waves can be reduced to one half by increasing the wave guide radius from 5 to 8 cm. For a 5 cm wave guide radius, 1 centimeter wavelength is close to the optimum.